ELSEVIER

Journal of Financial Markets 1 (1998) 321–352

Journal of FINANCIAL MARKETS

Strategic trading, asymmetric information and heterogeneous prior beliefs

F. Albert Wang*

Graduate School of Business, Columbia University, 420 Uris Hall, NY 10027, USA

Abstract

We develop a multi-period trading model in which traders face both asymmetric information and heterogeneous prior beliefs. Heterogeneity arises because traders agree to disagree on the precision of an informed trader's private signal. In equilibrium, the informed trader smooths out her trading on asymmetric information gradually over time, but concentrates her entire trading on heterogeneous beliefs toward the last few periods. As a result, the model's volume dynamics are consistent with the U-shaped intraday pattern at the close. Furthermore, the model predicts a positive autocorrelation in trading volume, and a positive correlation between trading volume and contemporaneous price volatility.

JEL classification: G12; G14

Keywords: Heterogeneous prior beliefs; Asymmetric information; Trading volume

1. Introduction

A casual observer of the U.S. securities markets would be amazed by the enormity of volume traded each day. For example, currently, the daily NYSE stock volume is, on average, about 500 million shares. What contributes to this enormous trading activity? Two motives for trade have been intensively explored in the literature: private information and liquidity (e.g., Grossman and Stiglitz, 1980; Kyle, 1985; Admati and Pfleiderer, 1988; Grundy and McNichols, 1989; Foster and Viswanathan, 1990, 1993; Holden and Subrahmanyam, 1992;

* Now with Rice University, Houston, TX, USA. E-mail: faw5@columbia.edu

PII S1386-4181(97)00007-4

Shalen, 1993; He and Wang, 1995). Kyle (1985) points out that liquidity trading provides camouflage which conceals informed trading and, as a result, the informed trading is swamped by liquidity trading. But, attributing the volume phenomenon to liquidity trading as a primary source is unappealing both theoretically and empirically. Thus, it is desirable to look for other important reasons for trading.

A third motive of trade that has attracted a lot of attention recently in the literature is heterogeneous prior beliefs (e.g., Harrison and Kreps, 1978; Varian, 1985, 1989; Hindy, 1989; De Long et al., 1990; Kandel and Pearson, 1992; Harris and Raviv, 1993). Heterogeneous prior beliefs arise either due to different priors about an asset's intrinsic value or due to non-concordant prior beliefs about the relationship between the signal and the value of the asset. Models of this kind do not depend on liquidity trading and hence do not have the aforementioned unsatisfactory property. But, the relative importance of this motive versus other motives in explaining the volume phenomenon is typically not addressed in these models. To deal with this problem, we develop a trading model that incorporates all three of the motives for trade and investigate the relative importance of each motive in terms of its contribution to total trading volume.

We extend the model of Kyle (1985) by incorporating heterogeneous prior beliefs. In our model, heterogeneity arises because traders have different distribution assumptions about an informed trader's private signal. In particular, traders agree to disagree with the precision of the signal. Our model thus captures Aumann's (1976) intuition of agreeing to disagree. Disagreements on the signal's precision may stem from the fallacy of excessive subjective certainty (Alpert and Raiffa, 1959; Kahneman and Tversky, 1982). In this view, a trader is *overconfident* if her distribution of the signal is too tight and *underconfident* if it is too loose; a trader is *rational* if her probability distribution reflects the correct one. Importantly, our analysis shows that divergence of beliefs cannot be unbounded for equilibrium to exist. Specifically, our model indicates that irrational beliefs cannot deviate from rational beliefs by more than a factor of two. For example, if a *rational* informed trader thinks a risky asset is worth \$100, then an *irrational* informed trader's subjective value cannot be less than \$0 or more than \$200.

Our model indicates that as noise trading vanishes in the limit, market makers will provide little liquidity while the informed trader will trade little. This result seems surprising because one might think that with little noise trading as camouflage, the informed trader and market makers, under risk neutrality, would agree to trade infinite quantities, given their differences in beliefs. This result underscores an important distinction between the case of noise trading vanishing in the limit and the case of no noise trading at all. Our model also shows that the speed of the price adjustment in each period does not depend on noise trading. Thus, even with little trading in the limit the price will still adjust, albeit partially, to the private signal's value over time.

The co-existence of asymmetric information and heterogeneous prior beliefs creates a dilemma in trading for the informed trader. On the one hand, to maximize the information advantage, the informed trader should refrain from aggressive trading; on the other hand, to exploit fully the differences in beliefs the informed trader should trade aggressively to enable the information asymmetry to vanish as quickly as possible. Our model shows that in equilibrium the informed trader is able to resolve the dilemma by trading on information first and on heterogeneous prior beliefs last. Furthermore, the model indicates that heterogeneous prior beliefs lead to a large increase in trading volume. In particular, informed trading can be larger than liquidity trading. These results suggest that heterogeneous prior beliefs may better explain the volume phenomenon than liquidity trading. Such a large increase in volume tends to concentrate toward the end, but not the beginning of trading. This result is consistent with the observed U-shaped intraday pattern in trading volume at the close (e.g., Wood et al., 1985; Jain and Joh, 1988; Gerety and Mulherin, 1992).

Our model yields other important implications for trading volume and price dynamics that are consistent with empirical evidence. The model predicts a positive correlation between trading volume and contemporaneous price volatility or absolute price changes. The model also predicts that heterogeneous prior beliefs lead to a positive autocorrelation in trading volume. The result that volume is serially correlated is also obtained by Foster and Viswanathan (1993), Wang (1994), Harris and Raviv (1993), and He and Wang (1995), and is consistent with the evidences in Harris (1987) and Campbell et al. (1993). The volume-volatility results are also obtained by Kim and Verrecchia (1991), Wang (1994), Shalen (1993), and Harris and Raviv (1993) and are documented in many empirical studies such as Karpoff (1987), Jain and Joh (1988), and Gallant et al. (1992). Our model contrasts with other models by the fact that these results are obtained from a trading model based on heterogeneous prior beliefs, rather than on diverse private information and that the heterogeneity is due to different interpretations of private information, not of public information.

Kyle and Wang (1997) consider heterogeneous prior beliefs in their study of the survival of irrational traders in a duopoly context. Wang (1997) examines the implication of overconfidence for delegated fund management in both learning and evolutionary game models. Our emphasis on the intertemporal dynamics of trading strategies and volume distinguishes our model from Wang (1997), and Kyle and Wang (1997).

This paper is organized as follows. Section 2 develops a sequential auction model and the unique equilibrium. The model's implications for trading volume and price dynamics are investigated in Section 3. Section 4 examines the properties of the equilibrium in detail including trading strategies, liquidity, price volatility, and trading volume. Section 5 concludes this paper. All proofs are contained in the appendix.

2. The model

A single risky asset with a terminal value $\tilde{v}$ is traded in an N-period sequential auction market among three kinds of risk-neutral traders: a monopolistic informed trader who has a unique access to a private signal $\tilde{s}^*$ about $\tilde{v}$; uninformed liquidity traders who trade randomly; and market makers who set prices efficiently (in the semi-strong form sense) conditional on the quantities traded by others.

2.1. Belief structure

The risky asset's terminal value $\tilde{v}$ is normally distributed with mean zero and variance Σ_0. All traders agree that the signal $\tilde{s}^*$ is a scalar multiple of the terminal value $\tilde{v}$, but they disagree concerning the right scale. Namely, the informed trader thinks $\tilde{s}^* = c_{\mathrm{i}}\tilde{v}$ while the market makers think $\tilde{s}^* = c_{\mathrm{m}}\tilde{v}$, where c_{i} and c_{m} are both positive scalars. It turns out that only the ratio of the two scalars matters in our analysis. Thus, without loss of generality, consider a normalized signal $\tilde{s} = \tilde{s}^*/c_{\mathrm{m}}$ in what follows. Then, the trader's beliefs can be rephrased as follows: the informed trader thinks $\tilde{s} = \tilde{v}/K$ while the market makers think $\tilde{s} = \tilde{v}$, where $K = c_{\mathrm{m}}/c_{\mathrm{i}}$ is a positive constant. Given this normalization, traders' heterogeneous prior beliefs are captured completely by the "belief" parameter K. The differences in beliefs may stem from the fallacy of excessive subjective certainty (see Alpert and Raiffa, 1959; Kahneman and Tversky, 1982). There is empirical evidence in psychology literature indicating that informed individuals tend to be overconfident, i.e., having unduly tight probability distributions (Oskamp, 1965). Consider the market makers' beliefs (i.e., $K = 1$) as the benchmark case, then the informed trader is 'overconfident' if her probability distribution of the signal $\tilde{s}$ is too 'tight' (i.e., $K > 1$) and 'underconfident' if it is too 'loose' (i.e., $0 < K < 1$).

2.2. Information structure

Trading takes place from time $t = 0$ to time $t = 1$. We denote by t_n the time at which the nth auction occurs for $n = 1, 2, \ldots, N$. Let $\tilde{x}_n$ be the aggregate position of the informed trader after the nth auction so that $\Delta\tilde{x}_n$ (defined by $\Delta\tilde{x}_n = \tilde{x}_n - \tilde{x}_{n-1}$) denotes the market order submitted by the informed trader at the nth auction, conditional on her information. At the first auction, the informed trader's information set contains only her observation of the private signal, while at the nth auction it also contains her observation of the past history of prices, $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{n-1}$. Let $\tilde{u}_n$ be the aggregate position of the liquidity traders after the nth auction so that $\Delta\tilde{u}_n$ (defined by $\Delta\tilde{u}_n = \tilde{u}_n - \tilde{u}_{n-1}$) denotes the random quantity traded by them at the nth auction. The process $\tilde{u}_n$ is assumed to be a Brownian motion and, hence, $\Delta\tilde{u}_n$ is normally distributed with zero mean and variance $\sigma_u^2 \Delta t_n$, and is independent of $\tilde{v}$. Thus, clearly, the liquidity traders' information set, if any, is irrelevant in our analysis. At each auction, market makers stand prepared to take a position in the asset to balance the demand of the remainder of the market. Thus, the order imbalance at the nth auction is $\Delta\tilde{y}_n$ (defined by $\Delta\tilde{y}_n = \Delta\tilde{x}_n + \Delta\tilde{u}_n$) and the market makers' information set at the nth auction contains the observations of all past and current order imbalances, $\Delta\tilde{y}_1, \Delta\tilde{y}_2, \ldots, \Delta\tilde{y}_n$.

2.3. The equilibrium

Let $\tilde{\pi}_n$ be the profit which accrues to the informed trader from the nth auction on,

$$\tilde{\pi}_n = \sum_{k=n}^{\mathrm{N}} (\tilde{v} - \tilde{p}_k)\Delta\tilde{x}_k, \quad \text{where } n = 1, \ldots, N. \tag{1}$$

Denote by $\mathrm{E}_K[\ldots]$ and $\mathrm{E}_1[\ldots]$ the expectation operators of the informed trader and the market makers, respectively. Given the above belief and information structures, at each auction the informed trader maximizes her total expected profits of the current and the remaining rounds of trading conditional on her information, i.e.,

$$\underset{\Delta x_n}{\mathrm{Max}} \ \mathrm{E}_K[\tilde{\pi}_n | \tilde{s} = s, \tilde{p}_1 = p_1, \ldots, \tilde{p}_{n-1} = p_{n-1}], \quad \text{for } n = 1, 2, \ldots, N. \tag{2}$$

We assume that competition forces market makers to set prices so that they earn zero expected profit conditional on their information at each auction, i.e.,

$$\tilde{p}_n = \mathrm{E}_1[\tilde{v} | \Delta\tilde{x}_1 + \Delta\tilde{u}_1, \ldots, \Delta\tilde{x}_n + \Delta\tilde{u}_n], \quad \text{for } n = 1, 2, \ldots, N. \tag{3}$$

The zero-profit condition implies that prices follow a martingale with respect to market makers' information set. Given this market efficiency condition, the amount of asymmetric information after the nth round of trading, denoted by Σ_n, is

$$\Sigma_n = \mathrm{E}_1[(\tilde{v} - \tilde{p}_n)^2] = \mathrm{Var}_1[\tilde{v} | \Delta\tilde{x}_1 + \Delta\tilde{u}_1, \ \ldots \ , \Delta\tilde{x}_n + \Delta\tilde{u}_n],$$
$$\text{for } n = 1, 2, \ldots, N. \tag{4}$$

As in Kyle (1985), this measure is called the *error variance of price* at time n. The zero-profit condition also implies that the price volatility, denoted by $\mathrm{Var}_1(\Delta\tilde{p}_n)$, is simply the amount of information revealed at the nth auction, i.e.,

$$\mathrm{Var}_1(\Delta\tilde{p}_n) = \Sigma_{n-1} - \Sigma_n. \tag{5}$$

Furthermore, we assume that market makers' pricing strategy is linear in their information set, $(\Delta\tilde{x}_1 + \Delta\tilde{u}_1, \ldots, \Delta\tilde{x}_n + \Delta\tilde{u}_n)$. Thus, equivalently, we consider a linear equilibrium in which the pricing rule, Δp_n, takes the following linear form:

$$\Delta\tilde{p}_n = \tilde{h}_{n-1} + \lambda_n(\Delta\tilde{x}_n + \Delta\tilde{u}_n), \quad \text{for } n = 1, 2, \ldots, N \tag{6}$$

where λ_n is some constant and $\tilde{h}_{n-1}$ is some linear function of $\Delta\tilde{x}_1 + \Delta\tilde{u}_1, \ldots, \Delta\tilde{x}_{n-1} + \Delta\tilde{u}_{n-1}$. In the equilibrium that emerges, this will be consistent with the zero-profit condition.

The informed trader cannot influence the pricing rule by committing to a particular trading rule before prices are established, while the market makers impute a trading strategy to the informed trader and set prices accordingly without observing the informed trader's quantity. Given the above structure, Theorem 1 below shows that the model has a unique linear equilibrium.

Theorem 1. *For* $0 < K < 2$, *there exists a unique linear recursive equilibrium. In this equilibrium there are constants* $\beta_n, \theta_n, \lambda_n, \gamma_n, \alpha_n, \omega_n, \phi_n, \delta_n$ *and* Σ_n *such that*

$$\Delta\tilde{x}_n = (1 + \gamma_n)\beta_n\Delta t_n(\tilde{s} - \tilde{p}_{n-1}) + \theta_n\Delta t_n\tilde{s}, \tag{7}$$

$$\Delta\tilde{p}_n = \gamma_n\tilde{p}_{n-1} + \lambda_n(\Delta\tilde{x}_n + \Delta\tilde{u}_n), \tag{8}$$

$$\Sigma_n = \mathrm{Var}_1[\tilde{v}|\Delta x_1 + \Delta u_1, \ldots, \Delta x_n + \Delta u_n], \tag{9}$$

$$\begin{aligned} &\mathrm{E}_K[\tilde{\pi}_n|\tilde{s} = s, \tilde{p}_1 = p_1, \ldots, \tilde{p}_{n-1} = p_{n-1}] \\ &\quad = \alpha_{n-1}(s - p_{n-1})^2 + \omega_{n-1}p_{n-1}s + \phi_{n-1}s^2 + \delta_{n-1}. \end{aligned} \tag{10}$$

Given Σ_0 *and* σ_u^2, *the above constants are the unique solution to the difference equation system as follows: for* $n = 1, 2, \ldots, N$,

$$\beta_n\Delta t_n = \frac{1 - 2\alpha_n\lambda_n}{2\lambda_n(1 - \alpha_n\lambda_n)}, \tag{11}$$

$$\theta_n\Delta t_n = \frac{-\gamma_n}{\lambda_n}, \tag{12}$$

$$\lambda_n = \frac{[(1 + \gamma_n)\beta_n + \theta_n]\Sigma_n}{\sigma_u^2}, \tag{13}$$

$$\gamma_n = 1 - K - \omega_n\lambda_n, \tag{14}$$

$$\Sigma_n = (1 + \gamma_n)(1 - \lambda_n\beta_n\Delta t_n)\Sigma_{n-1}, \tag{15}$$

$$\alpha_{n-1} = \frac{(1 + \gamma_n)^2}{4\lambda_n(1 - \alpha_n\lambda_n)}, \tag{16}$$

$$\omega_{n-1} = \omega_n + \frac{(2 - K)\gamma_n}{\lambda_n}, \tag{17}$$

$$\phi_{n-1} = \phi_n + \frac{-\gamma_n}{\lambda_n}, \tag{18}$$

$$\delta_{n-1} = \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n \tag{19}$$

subject to the boundary condition $\alpha_N = \omega_N = \phi_N = \delta_N = 0$ *and the second-order condition*

$$\lambda_n(1 - \alpha_n \lambda_n) > 0. \tag{20}$$

The inequality condition for the belief parameter K, i.e., $0 < K < 2$, results from the fact that the error variance of price is positive and strictly decreasing over time, i.e., $0 < \Sigma_n < \Sigma_{n-1}, \forall n$ (see Eq. (4)). To see this, combining Eq. (11) with Eq. (15) and rearranging yields $\Sigma_N / \Sigma_{N-1} = (2 - K)/2$ for the last period N. This leads to the inequality condition since $0 < \Sigma_N/\Sigma_{N-1} < 1 \Leftrightarrow 0 < K < 2$. Intuitively, the inequality condition means that if a *rational* informed trader thinks a risky asset is worth \$100, then an *irrational* informed trader's subjective value cannot be less than \$0 or more than \$200 for the equilibrium to exist. In addition, combining Eq. (13) with Eq. (15) and rearranging yields $1 - \lambda_n^2 \sigma_u^2 \Delta t_n / \Sigma_n = \Sigma_n / \Sigma_{n-1}$. This result implies the existence of a positive root of the liquidity parameter in each period, i.e., $\lambda_n > 0, \forall n$, and consequently the existence of the unique linear equilibrium, provided that the inequality condition is satisfied. What happens if the belief parameter K approaches its boundaries? In the case of extreme overconfidence (i.e., $K \to 2$), a simple calculation shows that both market liquidity and informed trading volume explode in the last period N, i.e., $\lambda_N \to 0$ and $\Delta \tilde{x}_n \to \infty$. In the case of extreme underconfidence (i.e., $K \to 0$), the informed trader's ex post expected value of the risky asset given her signal is always identical to the asset's ex ante common expected value, i.e., $\mathrm{E}_K[\tilde{v}|s] = \mathrm{E}_K[\tilde{v}] = \mathrm{E}_1[\tilde{v}]$. Thus, the informed trader lacks an incentive to trade and, as a result, her trading volume approaches zero uniformly for all periods, i.e., $\Delta \tilde{x}_n \to 0, \forall n$. Market makers recognize this fact and hence are willing to provide infinite liquidity, i.e., $\lambda_n \to 0, \forall n$, in this case.

Notice that the sequential auction equilibrium of Kyle (1985) corresponds to the common prior beliefs case of $K = 1$ in this equilibrium. Hence, by comparing this equilibrium with that of Kyle (1985), we are able to gauge the effect of heterogeneous prior beliefs. First, consider the informed trader's trading strategy, $\Delta \tilde{x}_n$. In Kyle's equilibrium, the strategy is simply a scalar multiple of $\tilde{s} - \tilde{p}_{n-1}$, the absolute value of which represents the informed trader's unit profit at the nth auction under common prior beliefs. The strategy in our model is a linear combination of $\tilde{s} - \tilde{p}_{n-1}$ and the size of the signal $\tilde{s}$, which is perfectly correlated with the perceived private value $(K - 1)\tilde{s}$. Thus, the first term in Eq. (7) represents the trading on information, while the second term represents the trading on heterogeneous prior beliefs.

Now, consider market maker's strategy, i.e., the pricing rule $\Delta \tilde{p}_n$. In Kyle's equilibrium, the pricing rule is simply a scalar multiple of the current order imbalance, $\Delta\tilde{x}_n + \Delta\tilde{u}_n$, which is the market makers' best forecast of the unobservable informed trading under common prior beliefs. The pricing rule in our model is a linear combination of the current order imbalance $\Delta\tilde{x}_n + \Delta\tilde{u}_n$ and the price of the previous period, $\tilde{p}_{n-1}$. This is so because the market makers know that part of the current order imbalance is induced by heterogeneous prior beliefs rather than information and, consequently, they use $\gamma_n\tilde{p}_{n-1}$ to undo this influence, as indicated in Eq. (8). Thus, an important testable implication of our model is the existence of a non-zero 'heterogeneity' parameter, γ_n, in the price innovation process of Eq. (8).

The parameters β_n and θ_n measure the intensity of the informed trading due to asymmetric information and heterogeneous prior beliefs, respectively. The parameters λ_n and γ_n characterize the pricing rule. In particular, the liquidity parameter, λ_n, is an inverse measure of market depth. The heterogeneity parameter, γ_n, measures the correction in the efficient price change per unit of the price in the previous period due to heterogeneous prior beliefs. The parameters α_{n-1}, ω_{n-1}, ϕ_{n-1}, and δ_{n-1} define the quadratic profit function of the informed trader at the n-th auction. The parameters α_{n-1} and δ_{n-1} correspond to the profit due to asymmetric information, whereas the parameters ω_{n-1} and ϕ_{n-1} correspond to the profit due to heterogeneous prior beliefs.

3. Equilibrium volume and price dynamics

Given the equilibrium of our model, in this section we investigate how heterogeneous prior beliefs affect the behavior of prices and trading volume. We show that (1) heterogeneous prior beliefs lead to a larger trading volume, (2) the volume of informed trading can be larger than that of liquidity trading, (3) heterogeneous prior beliefs lead to a serial correlation in volume, and (4) volume and contemporaneous price volatility or absolute price changes are positively correlated.

Following Admati and Pfleiderer (1988), the total trading volume at the n-th auction, denoted $\mathrm{Vol_n}$, is defined by

$$\mathrm{Vol}_n = \tfrac{1}{2}(|\Delta x_n| + |\Delta u_n| + |\Delta y_n|) \tag{21}$$

Given that $\Delta\tilde{x}_n$, $\Delta\tilde{u}_n$ and $\Delta\tilde{y}_n$ are normally distributed, each with zero mean, simple calculations yield the expected trading volume as shown in Proposition 1:

Proposition 1. *Heterogeneous prior beliefs lead to a larger trading volume and the expected trading volume at the nth auction is*

$$\mathrm{E}_1[\mathrm{Vol}_n] = \frac{1}{\sqrt{2\pi}} V_n^t = \frac{1}{\sqrt{2\pi}} (V_n^i + V_n^l + V_n^m), \tag{22}$$

where

$$V_n^i \equiv \sqrt{\mathrm{var}_1(\Delta\tilde{x}_n)} = \sqrt{\frac{(1-(\Sigma_n/\Sigma_{n-1}))}{\lambda_n^2}(\Sigma_{n-1}-\Sigma_n) + \frac{\gamma_n^2}{\lambda_n^2}(\Sigma_0-\Sigma_{n-1})}, \quad (23)$$

$$V_n^l = \sqrt{\mathrm{var}_1(\Delta\tilde{u}_n)} = \sigma_u\sqrt{\Delta t_n} = \sqrt{\frac{(\Sigma_n/\Sigma_{n-1})}{\lambda_n^2}(\Sigma_{n-1}-\Sigma_n)}, \quad (24)$$

$$V_n^m = \sqrt{\mathrm{var}_1(\Delta\tilde{y}_n)} = \sqrt{\frac{1}{\lambda_n^2}(\Sigma_{n-1}-\Sigma_n) + \frac{\gamma_n^2}{\lambda_n^2}(\Sigma_0-\Sigma_{n-1})}. \quad (25)$$

The above Proposition identifies the contribution of each group of traders to the total trading volume. Ignoring the scaling factor $1/\sqrt{2\pi}$, the expected total trading volume, V_n^t, is the sum of trades from the informed traders, V_n^i, the liquidity traders, V_n^l, and the market makers, V_n^m. Notice that the term $(\gamma_n^2/\lambda_n^2)(\Sigma_0-\Sigma_{n-1})$ in Eqs. (23) and (25) vanishes in the common prior beliefs case of $K=1$. Both the informed trader and the market makers trade a larger volume under heterogeneous prior beliefs and thus lead to an increase in total trading volume.

Informed trading consists of two sources: the term

$$\frac{(1-(\Sigma_n/\Sigma_{n-1}))}{\lambda_n^2}(\Sigma_{n-1}-\Sigma_n)$$

represents the volume generated by asymmetric information, whereas the other term $(\gamma_n^2/\lambda_n^2)(\Sigma_0-\Sigma_{n-1})$ represents the volume generated by heterogeneous prior beliefs. The volume of trading on information is proportional to $\Sigma_{n-1}-\Sigma_n$, which is exactly, by Eq. (5), the current price volatility $\mathrm{Var}_1(\Delta\tilde{p}_n)$. Thus, price volatility is indeed a good measure of information revelation through trading. But, volume of the trading on heterogeneous prior beliefs depends on the cumulative price volatility up to the last period, i.e., $\Sigma_0-\Sigma_{n-1}=\sum_{m=1}^{n-1}\mathrm{Var}_1(\Delta\tilde{p}_m)$. This result indicates a dilemma of informed trading. On the one hand, to exploit differences in beliefs, the informed trader should trade aggressively early on so that $\Sigma_0-\Sigma_{n-1}$ is large; on the other hand, to maximize her information advantage, she should restrict aggressive trading early on. It is therefore interesting to see how this dilemma is resolved in a dynamic context. In the next section, we show that the informed trader's optimal dynamic strategy facing such a dilemma can be characterized as follows: smooth trading on asymmetric information occurs continuously over time (so that information is revealed gradually); trading on beliefs is virtually nonexistent in all early auctions, but it increases dramatically and becomes substantial in the last few rounds.

Unlike the previous trading model based solely on asymmetric information and liquidity, informed trading in our model need not be dominated by liquidity trading. A straightforward comparison between Eqs. (23) and (24) yields the condition, in terms of the degree of heterogeneity, that leads to a larger volume of informed trading than liquidity trading, as shown in Proposition 2 below.

Proposition 2. *The expected volume of informed trading at the n-th auction is larger than that of liquidity trading if and only if*

$$\gamma_n^2 > \frac{[(2\Sigma_n/\Sigma_{n-1}) - 1](\Sigma_{n-1} - \Sigma_n)}{\Sigma_0 - \Sigma_{n-1}}, \quad \text{for } n = 1, 2, \dots, N. \tag{26}$$

The numerical results in the next section show that indeed the volume of informed trading can be larger than that of liquidity trading if the degree of heterogeneity is large enough, even though this occurs mainly in the last few auctions.

There is empirical evidence indicating that trading volume is autocorrelated (e.g., Harris, 1987; Campbell et al., 1993). Our model shows that heterogeneous prior beliefs may play a critical role in generating this result.

Proposition 3. *Informed trading on heterogeneous prior beliefs leads to an autocorrelation in order-imbalance and a positive autocorrelation in trading volume.*

Since liquidity trading is uncorrelated in our model, any autocorrelation in volume must come from informed trading. Note, however, that market efficiency condition implies that price changes are serially uncorrelated, i.e., $\mathrm{Cov}_1(\Delta\tilde{p}_n, \Delta\tilde{p}_{n+1}) = 0$. Under common prior beliefs, serially uncorrelated price changes, in turn, imply that market makers' order-imbalance and informed trading are both serially uncorrelated, i.e., $\mathrm{Cov}_1(\Delta\tilde{y}_n, \Delta\tilde{y}_{n+1}) = \mathrm{Cov}_1(\Delta\tilde{x}_n, \Delta\tilde{x}_{n+1}) = 0$. As a result, trading volume is serially uncorrelated under common prior beliefs. By contrast, under heterogeneous prior beliefs, serially uncorrelated price changes do not necessarily imply that order-imbalance and informed trading are serially uncorrelated. To see this, observe that Eq. (8) implies

$$\mathrm{Cov}_1(\Delta\tilde{y}_n, \Delta\tilde{y}_{n+1}) = \mathrm{Cov}_1\left(\frac{1}{\lambda_n}(\Delta\tilde{p}_n - \gamma_n\tilde{p}_{n-1}), \frac{1}{\lambda_{n+1}}(\Delta\tilde{p}_{n+1} - \gamma_{n+1}\tilde{p}_n)\right). \tag{27}$$

By Eq. (27), order-imbalance can be autocorrelated under heterogeneous prior beliefs, i.e., $\gamma_n \neq 0, \forall n$, despite the fact that price changes are still serially uncorrelated. In fact, as mentioned previously, informed trading on heterogeneous prior beliefs in each period depends on cumulative price volatility, i.e., $\Sigma_0 - \Sigma_{n-1} = \sum_{m=1}^{n-1}\mathrm{Var}_1(\Delta\tilde{p}_m)$, which by definition is overlapping over time. Consequently, correlated informed trading on heterogeneous prior beliefs generates an autocorrelation in order-imbalance and, ultimately, a positive autocorrelation in trading volume. Although informed trading on heterogeneous prior beliefs is unobservable, its effect can be gauged by market makers' order-imbalance autocorrelation, $\mathrm{Corr}_1(\Delta\tilde{y}_n, \Delta\tilde{y}_{n+1})$.

The result that volume exhibits serial correlation is also obtained by Foster and Viswanathan (1993), Wang (1994), Harris and Raviv (1993), and He and Wang (1995). Our model is distinguished from other models, except Harris and Raviv (1993), by attributing the sources of this serial correlation to trading on heterogeneous prior beliefs, not to trading on private information. Furthermore, the heterogeneity in Harris and Raviv's model is due to the different interpretations of public information, whereas in our model it is due to different interpretations of private information.

Many empirical studies document a positive correlation between volume and price volatility or absolute price changes over time (e.g., Karpoff, 1987; Jain and Joh, 1988; Gallant et al., 1992). It is obvious in our model that volume and contemporaneous price volatility is positively correlated since trading on information is determined by contemporaneous price volatility, as indicated in Eqs. (23) and (25). Similarly, the magnitudes of absolute price changes and of trading on information are both positively related to the amount of information revealed at each auction. Thus, our model also predicts a contemporaneous volume-absolute price changes correlation, as shown in Propositions 4.

Proposition 4. *The correlation between volume and contemporaneous price volatility or absolute price changes is positive.*

These results are similarly obtained by Kim and Verrecchia (1991), Wang (1994), Shalen (1993), and Harris and Raviv (1993). Our model differs from other models by showing that the result is obtained from a trading model based on heterogeneous prior beliefs, rather than on private information, where the heterogeneity is due to different interpretations of private information, not of public information.

4. General properties of the equilibrium

In Section 3, we focus on how heterogeneous prior beliefs affect the behavior of trading volume and prices. In this section, we assess the properties of the equilibrium with heterogeneous prior beliefs in more detail. To this end, following Holden and Subrahmanyam (1992), we develop an algorithm that analytically solves the model's unique equilibrium, as described in Theorem 2 below.

Theorem 2. *Let* $\Delta t_n = 1/N$, $q_n \equiv \alpha_n\lambda_n$ *and* $z_n \equiv \omega_n\lambda_n$, $\forall n = 1, 2, \ldots, N$.

For $0 < K < 2$, *the solution of the difference equation system in Theorem* 1 *is given by starting from the boundary condition* $q_N = z_N = 0$ *and iterating* *backward for* $q_{N-1}, \dots, q_1$ *and* $z_{N-1}, \dots, z_1$ *by using the unique root of the cubic equation as follows*:

$$q_{n-1}^3 - q_{n-1}^2 + a_n q_{n-1} + b_n = 0, \tag{28}$$

$$0 < q_{n-1} < 1, \tag{29}$$

$$a_n = \frac{(2-K-z_n)^2(z_n - K - 2q_n + 2Kq_n)}{4(K + z_n - 2q_n)}, \tag{30}$$

$$b_n = \frac{(2-K-z_n)^3 K}{8(K + z_n - 2q_n)}. \tag{31}$$

And, z_{n-1} *is given by*

$$z_{n-1} = \frac{4(1-K)(1-q_n)}{2-K-z_n} q_{n-1}. \tag{32}$$

Then starting from the exogenous values Σ_0 *and* σ_u^2, *iterate forward for the following variables in the order listed*:

$$\Sigma_n = \frac{(2-K-z_n)}{2(1-q_n)} \Sigma_{n-1}, \tag{33}$$

$$\lambda_n = \sqrt{\frac{(K + z_n - 2q_n)\Sigma_n}{2(1-q_n)\sigma_u^2 \Delta t_n}}, \tag{34}$$

$$\beta_n = \frac{(1-2q_n)}{2\lambda_n(1-q_n)\Delta t_n}, \tag{35}$$

$$\gamma_n = 1 - K - z_n, \tag{36}$$

$$\theta_n = \frac{-(1-K-z_n)}{\lambda_n \Delta t_n} \tag{37}$$

and

$$\alpha_{n-1} = \frac{(1+\gamma_n)^2}{4\lambda_n(1-q_n)}. \tag{38}$$

Finally, starting from the boundary condition $\omega_N = \phi_N = \delta_N = 0$, *iterate backward to calculate* ω_{n-1}, ϕ_{n-1} *and* δ_{n-1} *using Eqs.* (17)–(19) *respectively.*

The above algorithm utilizes three pivotal facts. First, the cubic Eq. (28) has a unique solution that satisfies the admissible range specified in Eq. (29). Second, Eqs. (28)–(32) are only functions of K and of the number of periods before the end of trading, i.e., $N - n + 1$, not of Δt_n. Starting from the boundary conditionq $q_N = z_N = 0$, the sequences $(q_{N-1}, q_{N-2}, \dots, q_1, q_0)$ *and* $(z_{N-1}, z_{N-2}, \dots, z_1, z_0)$ calculated backwards are the same regardless of what N is. Third, Eqs. (33)–(38)

show that all parameters in Theorem 1, i.e., β_n, θ_n, λ_n, γ_n, α_{n-1}, ω_{n-1}, ϕ_{n-1}, δ_{n-1}, and Σ_{n-1} are simply functions of q_n and z_n, $\forall n$. Therefore, given the boundary condition $\alpha_N = \omega_N = \phi_N = \delta_N = 0$, these parameters are uniquely determined once q_n and z_n are obtained using Eqs. (28)–(32).

4.1. Limiting results as noise trading vanishes

Given the algorithm above, we can examine the limiting result of our model as the variance of noise trading approaches zero, i.e., $\sigma_u^2 \to 0$. To do this, note that by Eq. (33) the terminal error variance of price is given by

$$\Sigma_N = \Sigma_0 \prod_{n=1}^{N} \frac{\Sigma_n}{\Sigma_{n-1}} = \Sigma_0 \prod_{n=1}^{N} \frac{2 - K - z_n}{2(1 - q_n)}.$$

By holding the number of period N fixed, q_n and z_n are uniquely determined such that $0 < \Sigma_n/\Sigma_{n-1} < 1$. Hence, not all private information is incorporated into prices by the end of trading with any finite number of period N, i.e., $\Sigma_N > 0$, $\forall N < \infty$. Applying these results to Eq. (34) and letting the variance of noise trading approach zero, the liquidity parameter approaches infinity, i.e., $\lambda_n \to \infty$ as $\sigma_u^2 \to 0$, $\forall n = 1, 2, \ldots, N$. Applying this limiting result to Eqs. (35) and (37), both intensity parameters approach zero, i.e., $\beta_n \to 0$ and $\theta_n \to 0$, $\forall n$. As a result, informed trading approaches zero uniformly as noise trading vanishes, i.e., $\Delta \tilde{x}_n \to 0$ as $\sigma_u^2 \to 0$, $\forall n$. This result seems surprising because one might think that with little noise trading as camouflage, the informed trader and market makers with risk neutrality would agree to trade infinite quantities, given their differences in beliefs. Our result indicates, on the contrary, that as noise trading vanishes in the limit, market makers will provide little liquidity while the informed trader will trade little. This result underscores an important distinction between the case of noise trading vanishing in the limit and the case of no noise trading at all. Interestingly, with such little trading in the limit the price will still adjust, albeit partially, to the private signal $\tilde{s}$'s value over time, and the adjustment does not depend on the size of noise trading. The complete limiting results of our model are stated in Proposition 5.

Proposition 5. *Given any finite number of period N, as the noise trading vanishes, i.e., $\sigma_u^2 \to 0$, the limiting results of our model are obtained as follows: for $\forall n$,*

$$\lambda_n \to \infty \text{ (market liquidity vanishes in the limit)}, \tag{39}$$

$$\beta_n \to 0 \text{ and } \theta_n \to 0 \text{ (informed trading intensities vanish)}, \tag{40}$$

$$\alpha_{n-1} \to 0,\ \omega_{n-1} \to 0,\ \phi_{n-1} \to 0, \text{ and } \delta_{n-1} \to 0 \text{ (all profit parameters vanish)}, \tag{41}$$

$$\Delta \tilde{x}_n \to 0 \text{ (informed trading volume vanishes)}, \tag{42}$$

$$\Delta \tilde{p}_n \to \Delta \tilde{p}_n = c_n(\tilde{s} - \tilde{p}_{n-1}) + (c_n \Sigma_n)^{1/2} \tilde{\xi}_n, \text{ (partial adjustment of the price)}, \tag{43}$$

where $c_n = 1 - \Sigma_n/\Sigma_{n-1}$ *is a constant such that* $0 < c_n < 1$ *and* $\tilde{\xi}_n \sim \mathrm{N}(0, 1)$.

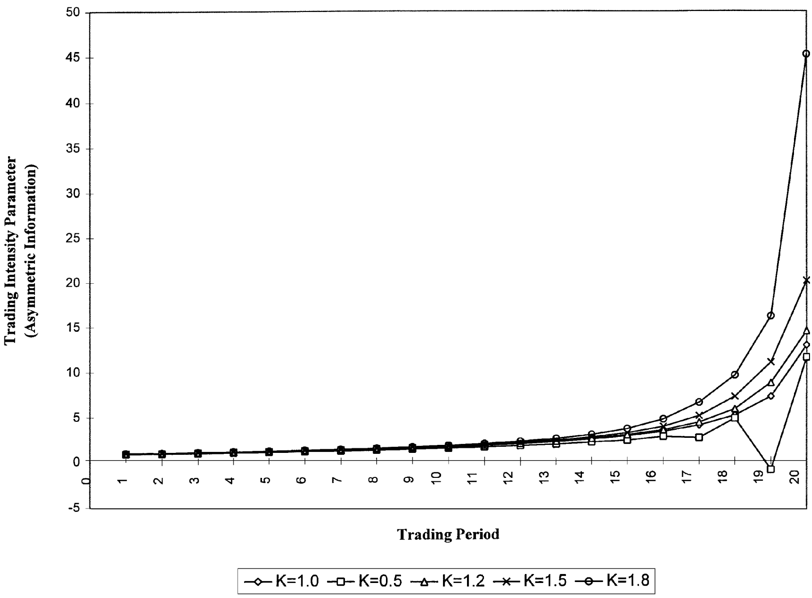
The above limiting results indicate that informed trading and profits are both proportional to noise trading; the private information is partial incorporated into prices over time and, clearly, the price adjustment does not depend on the size of noise trading. In this sense, as in Kyle (1985), noise trading does not destabilize the market and, on the contrary, it enhances market liquidity and facilitates trades.

4.2. Numerical results for different degrees of heterogeneity

In order to compare the heterogeneous priors case to the common priors case, we use the above algorithm to generate a series of numerical results of our model by holding constants $N = 20$, $\Sigma_0 = 1$ and ${\sigma_u}^2 = 1$ fixed and varying the belief parameter $K = 0.5$, 1, 1.2, 1.5, or 1.8. Notice that the informed trader is overconfident (underconfident) if $K > 1$ ($K < 1$) while the case of $K = 1$ is used as the benchmark case in which all traders have common prior beliefs, as in Kyle (1985). We examine the dynamic behavior of the trading intensity parameters of asymmetric information and heterogeneous beliefs, β_n and θ_n, respectively, the liquidity parameter, λ_n, the heterogeneity parameter, γ_n, the error variance of price, Σ_n, the price volatility, $\mathrm{Var}_1(\Delta \tilde{p}_n)$, the expected total trading volume, V_n^t, and its breakdown into various groups of traders: the informed trader, V_n^i, the liquidity traders, V_n^l, and the market makers, V_n^m, and the order-imbalance autocorrelation, $\mathrm{Corr}_1(\Delta \tilde{y}_n, \Delta \tilde{y}_{n+1})$.

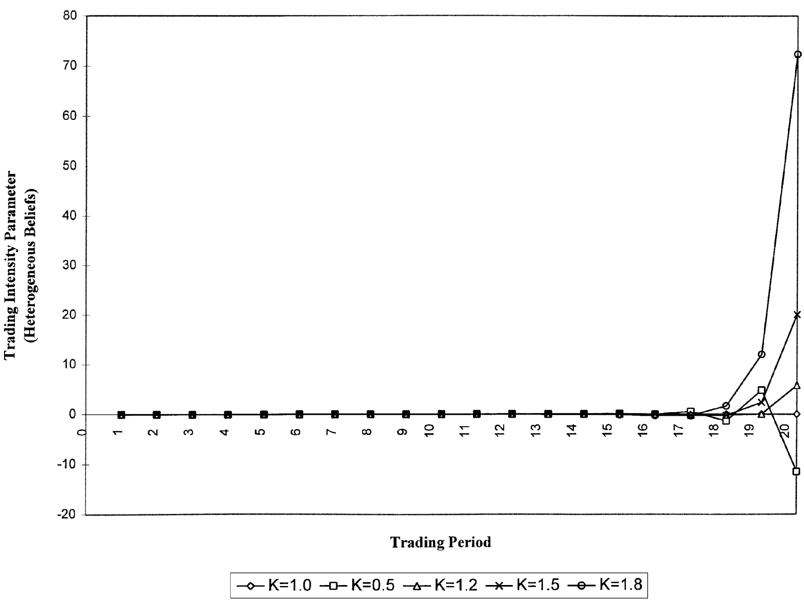
Fig. 1 indicates that the informed trader restricts her intensity of trading on information, measured by β_n, during all early auctions and becomes aggressive only in the last few rounds of trades. The aggressiveness of trading on information in those last few rounds rises with the confidence level, i.e., the belief parameter, K. This result is intuitive since the more confident the informed trader is about her information the more aggressively she will trade on the information. Fig. 2 shows that the informed trader trades on differences in beliefs only in the last few auctions since the intensity of trading on heterogeneous prior beliefs, measured by θ_n, is negligible in all early auctions. The aggressiveness of trading on heterogeneous prior beliefs in those last few rounds rises with the degree of heterogeneity, measured by $|K - 1|$.

Fig. 3 shows that the liquidity parameter, λ_n, remains high and flat during all early auctions and it drops dramatically in the last few ones, since by then most information asymmetry is resolved through trading. This pattern is consistent with the informed trader's strategies as indicated in Figs. 1 and 2. The informed



50
45
40
35
30
25
20
15
10
5
0
-5
Trading Intensity Parameter (Asymmetric Information)
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Trading Period
K=1.0 K=0.5 K=1.2 K=1.5 K=1.8

Fig. 1.



80
70
60
50
40
30
20
10
0
-10
-20
Trading Intensity Parameter (Heterogeneous Beliefs)
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Trading Period
K=1.0 K=0.5 K=1.2 K=1.5 K=1.8

Fig. 2.

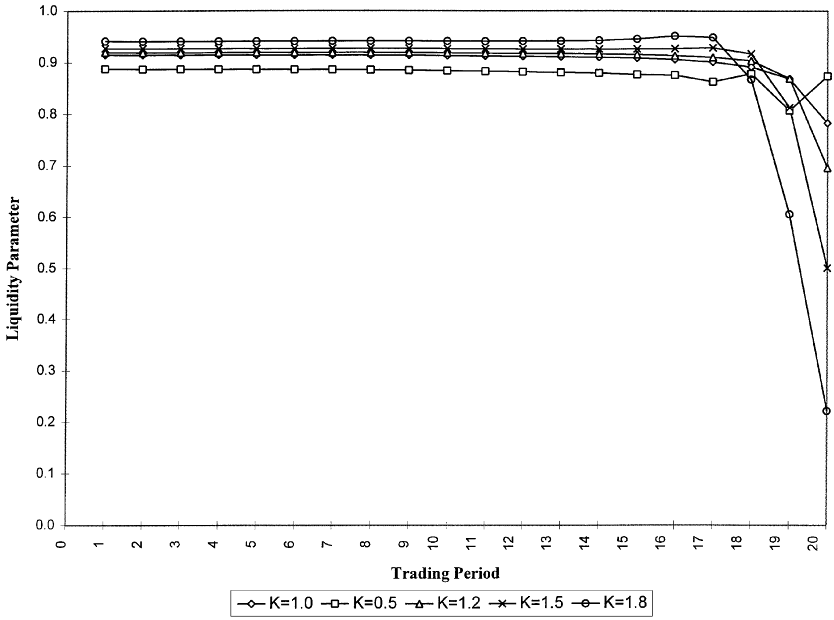


Fig. 3.

trader restricts her trading in early auctions when market liquidity remains poor and accelerates her trading in the last few rounds when the liquidity is much better. Market liquidity is worse for more aggressive trading, measured by a larger K, during all early auctions because of a worse adverse selection problem. The situation reverses in the last few rounds, because there is less information asymmetry, measured by Σ_n, present in the market.

Fig. 4 shows that the heterogeneity parameter, γ_n, is almost nil in all early auctions and becomes significant only in the last few ones. This pattern is again consistent with the informed trader's strategies. Market makers correctly predict that heterogeneous prior beliefs have an impact on informed trading only in the last few auctions and, consequently, undo the impact in those last few auctions, indicated by a non-zero γ_n, in order to account for the adverse selection problem properly. This intuition is further confirmed by the fact that the pattern in Fig. 4 is exactly the opposite of the pattern in Fig. 2.

Fig. 5 indicates that less information asymmetry, measured by the error variance of price Σ_n, is present in the market with more aggressive informed trading, measured by the belief parameter K. Fig. 6 shows that indeed the informed trader is modest in her trading early on and becomes aggressive only in the last few rounds. Hence, the amount of information revealed, measured by the price volatility $\mathrm{Var}_1(\Delta\tilde{p}_n)$, is typically larger in the last few auctions. The price

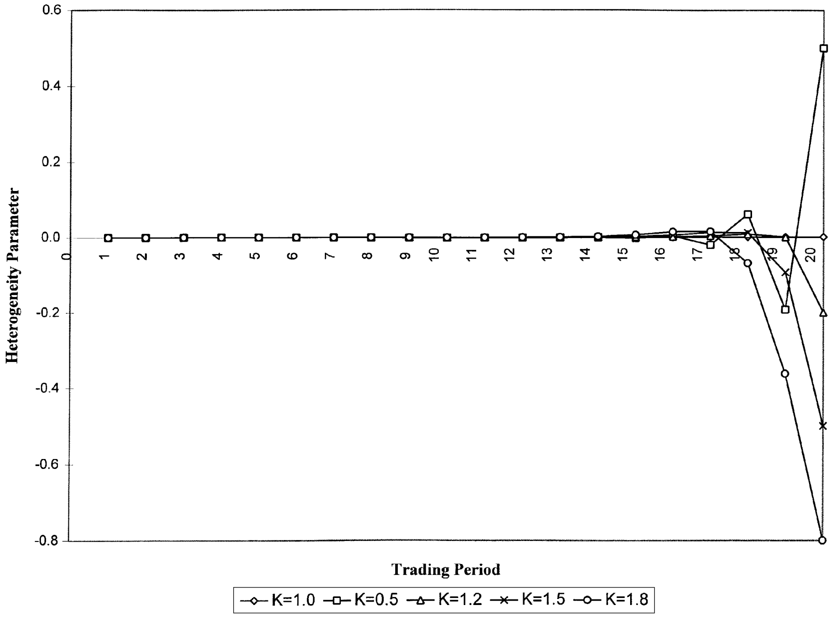

Heterogeneity Parameter
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Trading Period
K=1.0 K=0.5 K=1.2 K=1.5 K=1.8

Fig. 4.

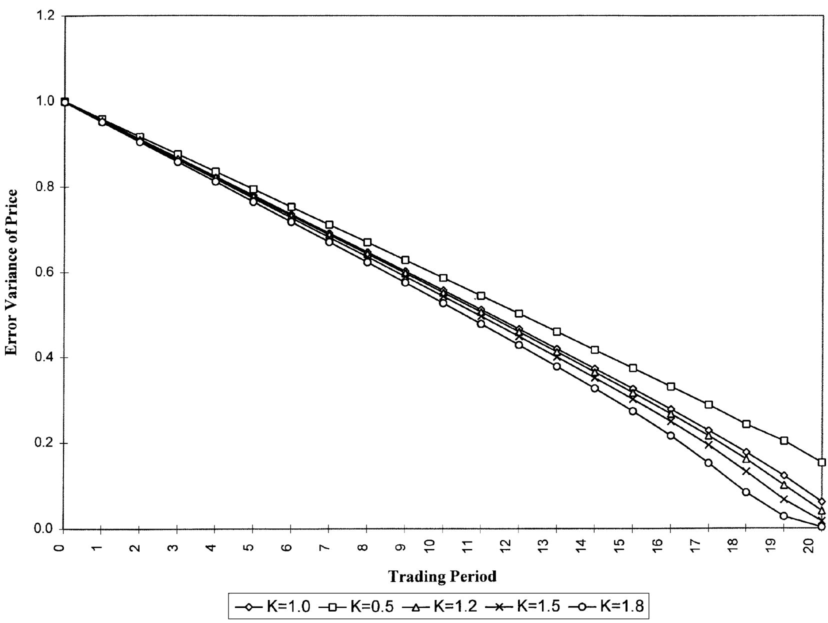

Error Variance of Price
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Trading Period
K=1.0 K=0.5 K=1.2 K=1.5 K=1.8

Fig. 5.

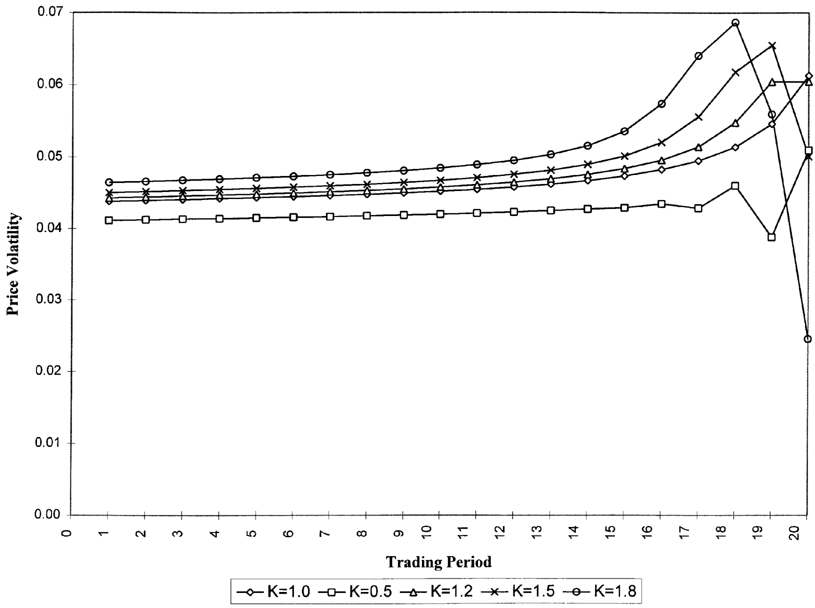


Fig. 6.

volatility generally rises with the level of confidence, measured by the belief parameter K.

Fig. 7 shows that the total expected trading volume, V^t_n, remains flat during all early auctions and jumps significantly in the last few ones when aggressive trading on heterogeneous prior beliefs occurs. This result vividly confirms Proposition 1 that heterogeneous prior beliefs lead to a larger trading volume. Moreover, the dynamic pattern in Fig. 7 indicates that such a larger volume occurs mainly in the last few auctions and is positively related to the degree of heterogeneity, measured by $|K-1|$.

A closer look at the breakdown of the total expected volume for the case of $K=1.5$ in Fig. 8 indicates that informed trading, V^i_n, becomes larger than that of the liquidity trading, V^l_n, near the end of trading. This result confirms Proposition 2 that informed trading can be larger than liquidity trading under heterogeneous prior beliefs. The results from Figs. 7 and 8 suggest that heterogeneous prior beliefs magnify trading volume and may better explain the high volume phenomenon in financial markets than the liquidity trading proposed in previous trading models under common prior beliefs. Finally, the dynamic pattern in Fig. 7 exhibits a concentration pattern near the end of trading. This pattern is consistent with the widely documented U-shaped intraday pattern in trading volume at the close (Wood et al., 1985; Jain and Joh, 1988).

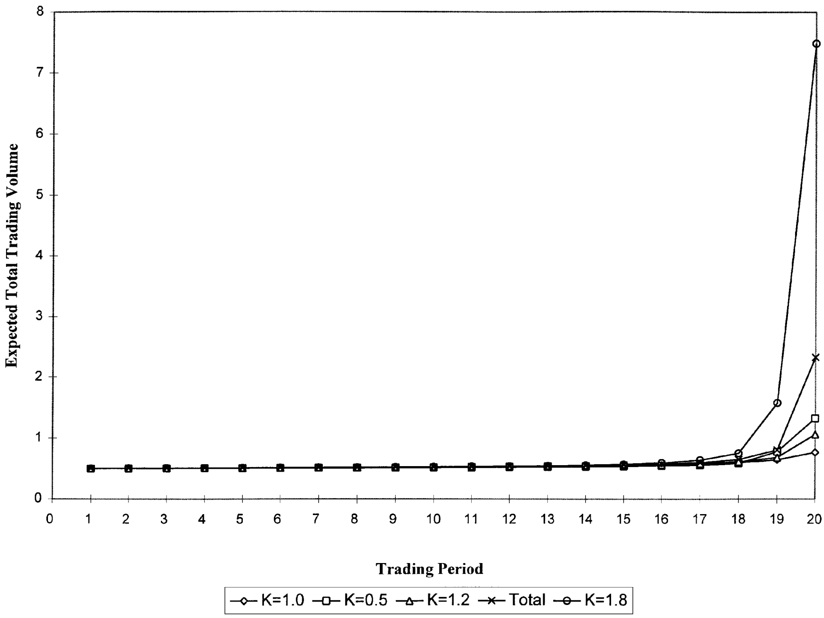

Expected Total Trading Volume
Trading Period
K=1.0 K=0.5 K=1.2 Total K=1.8

Fig. 7.

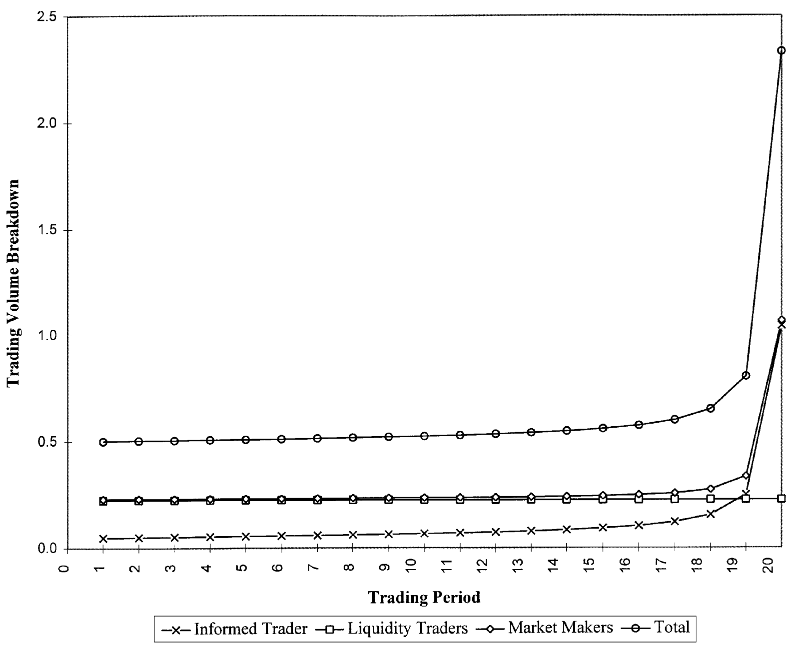

Trading Volume Breakdown
Trading Period
Informed Trader Liquidity Traders Market Makers Total

Fig. 8.

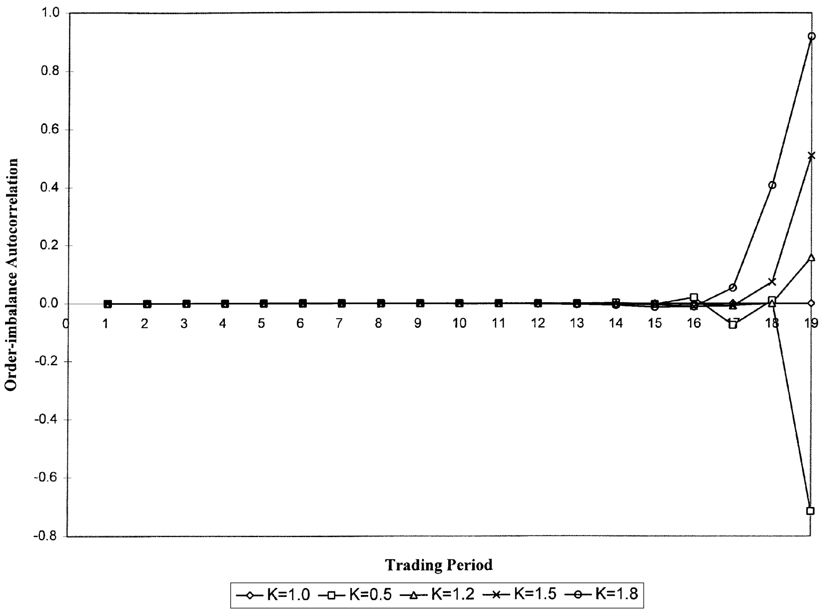


Fig. 9.

Fig. 9 indicates that the order-imbalance autocorrelation, $\mathrm{Corr}_1(\Delta\tilde{y}_n, \Delta\tilde{y}_{n+1})$, is non-zero only in the last few auctions. This implies that trading on heterogeneous prior beliefs is significant in those periods, which leads to a positive autocorrelation in volume. The trading volume in all early auctions is nevertheless uncorrelated. This result confirms Proposition 3 that such a positive autocorrelation in volume is driven by aggressive trading on heterogeneous prior beliefs.

To ensure that the above findings do not depend on the particular number of auctions chosen, i.e., $N = 20$, we also study the numerical results for the cases of $N = 4$, 100, and 1000. In all cases, we obtain essentially the same insights regarding the effects of heterogeneous prior beliefs in our model.

5. Conclusions

We have investigated a dynamic model of strategic trading when traders face both asymmetric information and heterogeneous prior beliefs, in additional to liquidity shocks. In the model, heterogeneous prior beliefs arise because traders agree to disagree with the precision of an informed trader's private signal. But, the divergence of beliefs cannot be unbounded for equilibrium to exist. Our model thus captures Aumann's (1976) intuition of *agreeing to disagree* and contrasts with previous trading models with common prior beliefs.

Our model indicates that as noise trading vanishes in the limit, market makers will provide little liquidity while the informed trader will trade little. This result seems surprising because one might think that with little noise trading as camouflage, the informed trader and market makers with risk neutrality would agree to trade infinite quantities, given their differences in beliefs. Our model also shows that the speed of price adjustment in each period does not depend on noise trading. Thus, even with little trading in the limit the price will adjust, albeit partially, to the signal's value over time.

Our model implies that in equilibrium a separation of trading motives emerges: the motive of asymmetric information dominates in early rounds of trading, whereas heterogeneous prior beliefs prevail in the last few rounds. In particular, the model shows that heterogeneous prior beliefs result in a large increase in trading volume, and hence may better explain the volume phenomenon in financial markets than liquidity trading. Such a large increase in volume tends to concentrate toward the end of trading. This result is consistent with the widely documented U-shaped intraday pattern in trading volume at the close (e.g., Wood et al., 1985; Gerety and Mulherin, 1992).

Our model predicts a positive correlation between trading volume and contemporaneous price volatility or absolute price changes. The model also predicts that heterogeneous prior beliefs lead to a positive autocorrelation in volume. These results are consistent with empirical evidence (e.g., Harris, 1987; Karpoff, 1987; and Gallant et al., 1992). Our model contrasts with other models with similar predictions by the fact that these results are obtained from a trading model based on heterogeneous prior beliefs, not on diverse private information (e.g., Kim and Verrecchia, 1991; He and Wang, 1995). Furthermore, the heterogeneity in our model is due to different interpretations of private information, not of public information (e.g. Harris and Raviv, 1993).

Acknowledgements

I am grateful to Pete Kyle for invaluable guidance. I thank Utpal Bhattacharya, Larry Glosten, Craig Holden, Vincente Madrigal, A. Subrahmanyam (the editor), S. Viswanathan, and Ho-Mou Wu for helpful discussions and comments. I have also received helpful comments from seminar participants at Columbia University, New York University, and the 1995 Western Finance Association Meetings.

Appendix

Proof of Theorem 1. Using a dynamic programming approach, we first hypothesize that the expected profit of the informed trader from the $n+1$th auction on is, for $n = 0, 1, 2, \dots, N$,

$$\mathrm{E}_K[\tilde{\pi}_{n+1}|\tilde{s} = s, \tilde{p}_1 = p_1, \dots, \tilde{p}_n = p_n] = \alpha_n(s - p_n)^2 + \omega_n p_n s + \phi_n s^2 + \delta_n. \tag{A.1}$$

Obviously, this hypothesis satisfies the boundary condition for $\alpha_N = \omega_N = \phi_N = \delta_N = 0$ since no profits can be made after trading is completed. At the nth auction, the informed trader chooses an optimal trading quantity Δx_n in order to maximize her expected profit from the nth auction on, given her observation of the private signal and past prices, i.e.,

$$\underset{\Delta x_n}{\mathrm{Max}}\ \mathrm{E}_K[\tilde{\pi}_n|\tilde{s} = s, \tilde{p}_1 = p_1, \dots, \tilde{p}_{n-1} = p_{n-1}]. \tag{A.2}$$

Since $\tilde{\pi}_n$ is given recursively by $\tilde{\pi}_n = (\tilde{v} - \tilde{p}_n)\Delta\tilde{x}_n + \tilde{\pi}_{n+1}$, we can rewrite Eq. (A.2) as

$$\underset{\Delta x_n}{\mathrm{Max}}\ \mathrm{E}_K[(\tilde{v} - \tilde{p}_n)\Delta\tilde{x}_n + \alpha_n(\tilde{s} - \tilde{p}_n)^2 + \omega_n p_n \tilde{s} + \phi_n \tilde{s}^2 + \delta_n | s, p_1, \dots, p_{n-1}]. \tag{A.3}$$

In a linear equilibrium, substituting the expression of $\tilde{p}_n$ from Eq. (6) into Eq. (A.3) and evaluating the conditional expectation yields

$$\begin{aligned}\underset{\Delta x_n}{\mathrm{Max}}\ &\mathrm{E}_K[\tilde{\pi}_n|\tilde{s} = s, \tilde{p}_1 = p_1, \dots, \tilde{p}_{n-1} = p_{n-1}]\\ &= \underset{\Delta x_n}{\mathrm{Max}}\ [(\mathrm{K}s - p_{n-1} - h_{n-1} - \lambda_n\Delta x_n)\Delta x_n + \alpha_n(s - p_{n-1} - h_{n-1} - \lambda_n \Delta x_n)^2\\ &\quad + \alpha_n\lambda_n^2\sigma_u^2\Delta t_n + \omega_n(p_{n-1} + h_{n-1} + \lambda_n\Delta x_n)s + \phi_n s^2 + \delta_n].\end{aligned} \tag{A.4}$$

The first-order condition of Eq. (A.4) yields the optimal trading strategy

$$\Delta x_n = \beta_n\Delta t_n(s - p_{n-1} - h_{n-1}) + \psi_n\Delta t_n s \quad \forall n = 1, 2, \dots, N, \tag{A.5}$$

where $\beta_n\Delta t_n$ is given by Eq. (11) and $\psi_n\Delta t_n$ is given by

$$\psi_n\Delta t_n = \frac{K - 1 + \omega_n\lambda_n}{2\lambda_n(1 - \alpha_n\lambda_n)}. \tag{A.6}$$

We can identify the exact linear form of $\tilde{h}_{n-1}$ using the fact that $\tilde{p}_n$ is a martingale with respect to the corresponding information set $(\Delta\tilde{x}_1 + \Delta\tilde{u}_1,$

$\dots, \Delta\tilde{x}_n + \Delta\tilde{u}_n)$. Thus, we obtain

$$\mathrm{E}_1[\Delta p_n | \Delta\tilde{x}_1 + \Delta\tilde{u}_1, \dots, \Delta\tilde{x}_{n-1} + \Delta\tilde{u}_{n-1}] = 0 \quad \forall n = 1, \dots, N. \tag{A.7}$$

Substituting $\Delta\tilde{p}_n$ from Eq. (6) into Eq. (A.7) yields

$$\mathrm{E}_1[\tilde{h}_{n-1} + \lambda_n(\Delta\tilde{x}_n + \Delta\tilde{u}_n) | \Delta\tilde{x}_1 + \Delta\tilde{u}_1, \dots, \Delta\tilde{x}_{n-1} + \Delta\tilde{u}_{n-1}] = 0$$
$$\forall n = 1, \dots, N. \tag{A.8}$$

Substituting $\Delta\tilde{x}_n$ from Eq. (A.5) into Eq. (A.8) and evaluating the conditional expectation yields

$$(1 - \lambda_n \beta_n \Delta t_n) h_{n-1} + \lambda_n \psi_n \Delta t_n p_{n-1} = 0 \quad \forall n = 1, \dots, N. \tag{A.9}$$

Substituting $\beta_n \Delta t_n$ and $\psi_n \Delta t_n$ from Eqs. (11) and (A.6) into Eq. (A.9) and rearranging yields

$$h_{n-1} = \gamma_n p_{n-1} \quad \forall n = 1, \dots, N, \tag{A.10}$$

where γ_n is given by Eq. (14). Plugging Eq. (A.10) into Eq. (A.5) and rearranging yields the ex ante optimal trading strategy:

$$\Delta\tilde{x}_n = (1 + \gamma_n)\beta_n \Delta t_n (\tilde{s} - \tilde{p}_{n-1}) + \theta_n \Delta t_n \tilde{s} \quad \forall n = 1, \dots, N, \tag{A.11}$$

where $\theta_n \Delta t_n$ is given by Eq. (12). Similarly, plugging Eq. (A.10) into Eq. (6) yields the ex ante efficient pricing rule:

$$\Delta\tilde{p}_n = \gamma_n \tilde{p}_{n-1} + \lambda_n(\Delta\tilde{x}_n + \Delta\tilde{u}_n) \quad \forall n = 1, \dots, N. \tag{A.12}$$

To identify λ_n, we use again the fact that $\tilde{p}_n$ is a martingale and thus obtain

$$\Delta\tilde{p}_n = \mathrm{E}_1[\tilde{v} - \tilde{p}_{n-1} | \Delta\tilde{x}_1 + \Delta\tilde{u}_1, \dots, \Delta\tilde{x}_n + \Delta\tilde{u}_n] \quad \forall n = 1, \dots, N. \tag{A.13}$$

By the projection theorem, Eqs. (A.12) and (A.13) together imply that $\Delta\tilde{p}_n$ is the projection of $\tilde{v} - \tilde{p}_n$ onto the n-dimension space spanned by $(\Delta\tilde{x}_1 + \Delta\tilde{u}_1, \dots, \Delta\tilde{x}_n + \Delta\tilde{u}_n)$ in which λ_n is the coefficient of the nth orthogonal basis, $(\gamma_n/\lambda_n)\tilde{p}_{n-1} + \Delta\tilde{x}_n + \Delta\tilde{u}_n$. By the martingale property, $\tilde{v} - \tilde{p}_n$ is orthogonal to all other $n - 1$ bases. Thus, we may rewrite Eq. (A.13) as

$$\Delta\tilde{p}_n = \mathrm{E}_1\left[\tilde{v} - \tilde{p}_{n-1} \middle| \frac{\gamma_n}{\lambda_n}\tilde{p}_{n-1} + \Delta\tilde{x}_n + \Delta\tilde{u}_n\right] \quad \forall n = 1, \dots, N. \tag{A.14}$$

Substituting $\Delta\tilde{x}_n$ from Eq. (A.11) into Eq. (A.14) and evaluating the expectation, we obtain

$$\lambda_n = \frac{[(1 + \gamma_n)\beta_n \Delta t_n + \theta_n \Delta t_n]\Sigma_{n-1}}{[(1 + \gamma_n)\beta_n \Delta t_n + \theta_n \Delta t_n]^2 \Sigma_{n-1} + \sigma_u^2 \Delta t_n} \quad \forall n = 1, \dots, N. \tag{A.15}$$

Using Eqs. (A.11) and (A.12) and calculating explicitly Σ_n in Eq. (4), we obtain

$$\Sigma_n = \frac{\Sigma_{n-1}\sigma_u^2 \Delta t_n}{[(1+\gamma_n)\beta_n \Delta t_n + \theta_n \Delta t_n]^2 \Sigma_{n-1} + \sigma_u^2 \Delta t_n} \quad \forall n = 1, \ldots, N. \tag{A.16}$$

A simple calculation shows that Eqs. (A.15) and (A.16) together imply Eqs. (13) and (15).

To verify that our hypothesis for the expected profit in Eq. (A.1) is consistent with the equilibrium, we plug Eqs. (A.10) and (A.11) into Eq. (A.4) and rearrange terms. Thus, we obtain

$$\begin{aligned} &\mathrm{E}_K[\tilde{\pi}_n | \tilde{s} = s, \tilde{p}_1 = p_1, \ldots, \tilde{p}_{n-1} = p_{n-1}] \\ &\quad = \alpha_{n-1}(s - p_{n-1})^2 + \omega_{n-1} p_{n-1} s + \phi_{n-1} s^2 + \delta_{n-1}, \end{aligned} \tag{A.17}$$

where α_{n-1}, ω_{n-1}, ϕ_{n-1} and δ_{n-1} are given by Eqs. (16)–(19), respectively. Obviously, Eq. (A.1) is consistent with Eq. (A.17) in equilibrium.

So far, it is shown that, given Σ_0 and σ_u^2, Eqs. (11)–(20) and the boundary condition $\alpha_N = \omega_N = \phi_N = \delta_N = 0$ are the necessary and sufficient conditions for a recursive linear equilibrium $\Delta \tilde{x}_n$ and $\Delta \tilde{p}_n$ as in Eqs. (7) and (8), respectively. We now prove the uniqueness of this linear equilibrium by showing that the difference equation system Eqs. (11)–(19) has a unique solution satisfying the boundary condition $\alpha_N = \omega_N = \phi_N = \delta_N = 0$ and the second-order condition (20). Use Eqs. (11)–(14) together and simplify to get

$$\left(1 - \frac{\lambda_n^2 \sigma_u^2 \Delta t_n}{\Sigma_n}\right)(1 - \alpha_n \lambda_n) = \frac{2 - K - \omega_n \lambda_n}{2} \quad \forall n = 1, 2, \ldots, N. \tag{A.18}$$

Note, Eqs. (16) and (20) together imply

$$\alpha_{n-1} > 0 \quad \forall n = 1, 2, \ldots, N, \tag{A.19}$$

$$\lambda_n > 0 \quad \forall n = 1, 2, \ldots, N. \tag{A.20}$$

Applying the boundary condition $\alpha_N = \omega_N = 0$ to (A.18) yields

$$1 - \frac{\lambda_N^2 \sigma_u^2 \Delta t_N}{\Sigma_N} = \frac{2 - K}{2}. \tag{A.21}$$

Given $K > 0$, from Eqs. (A.20) and (A.21), we obtain the boundary condition for any finite N,

$$\Sigma_N > 0. \tag{A.22}$$

Given Eq. (A.22) and the market efficiency condition, we have

$$0 < \Sigma_n < \Sigma_{n-1} \quad \forall n = 1, 2, \ldots, N. \tag{A.23}$$

Combining Eq. (13) with Eq. (15) and rearranging yields

$$1 - \frac{\lambda_n^2 \sigma_u^2 \Delta t_n}{\Sigma_n} = \frac{\Sigma_n}{\Sigma_{n-1}}. \tag{A.24}$$

From Eqs. (A.20), (A.23) and (A.24), we obtain

$$0 < \frac{\lambda_n^2 \sigma_u^2 \Delta t_n}{\Sigma_n} < 1 \quad \forall n = 1, 2, \ldots, N. \tag{A.25}$$

Note that Eqs. (A.21) and (A.25) together yields the inequality condition

$$0 < K < 2. \tag{A.26}$$

Given α_n, ω_n, Σ_n, and the inequality condition $0 < K < 2$, Eq. (A.18) is a cubic equation in λ_n which has three roots. But only the middle root satisfies the second-order condition, the equilibrium λ_n is thus uniquely determined. Furthermore, given $\alpha_n, \omega_n, \phi_n, \delta_n, \Sigma_n$ and λ_n, it is straightforward to obtain $\gamma_n, \beta_n, \theta_n, \alpha_{n-1}, \omega_{n-1}, \phi_{n-1}, \delta_{n-1}$ and Σ_{n-1} from the difference equation system (11)–(19). Thus we have iterated the system backwards one step.

Given the inequality condition $0 < K < 2$, the second-order condition $\lambda_n(1 - \alpha_n \lambda_n) > 0$ and the boundary condition $\alpha_N = \omega_N = \phi_N = \delta_N = 0$, the backward iteration procedure yields a family of solutions to the difference equation system parameterized by the terminal value Σ_N. Moreover, if $\lambda_n, \gamma_n, \beta_n, \theta_n, \alpha_{n-1}, \omega_{n-1}, \phi_{n-1}, \delta_{n-1}$ and Σ_{n-1} solve the difference equation system given arbitrary Σ_N and $\alpha_N = \omega_N = \phi_N = \delta_N = 0$, then for any positive constant c, it is true that $c\lambda_n, \gamma_n, (1/c)\beta_n, (1/c)\theta_n, (1/c)\alpha_{n-1}, (1/c)\omega_{n-1}, (1/c)\phi_{n-1}$, $c\delta_{n-1}$ and $c^2\Sigma_{n-1}$ also solve the difference equation system with terminal value $c^2\Sigma_N$. Thus, Σ_N is proportional to Σ_0 in any solution and consequently there is a unique solution given the initial value Σ_0. Q.E.D.

Proof of Proposition 1. Taking the expectation on both sides of Eq. (21) and recognizing that $\Delta\tilde{x}_n, \Delta\tilde{u}_n$ and $\Delta\tilde{y}_n$ are normally distributed, each with zero mean, we obtain

$$\begin{aligned} \mathrm{E}_1[\mathrm{Vol}_n] &= \tfrac{1}{2}(\mathrm{E}_1[|\Delta\tilde{x}_n|] + \mathrm{E}_1[|\Delta\tilde{u}_n|] + \mathrm{E}_1[|\Delta\tilde{y}_n|]) \\ &= \frac{1}{\sqrt{2\pi}}(\sqrt{\mathrm{var}_1(\Delta\tilde{x}_n)} + \sqrt{\mathrm{var}_1(\Delta\tilde{u}_n)} + \sqrt{\mathrm{var}_1(\Delta\tilde{y}_n)}) \\ &= \frac{1}{\sqrt{2\pi}}(V_n^i + V_n^l + V_n^m) = \frac{1}{\sqrt{2\pi}} V_n^t, \end{aligned} \tag{A.27}$$

where $V^i_n \equiv \sqrt{\mathrm{var}_1(\Delta\tilde{x}_n)}$, $V^l_n \equiv \sqrt{\mathrm{var}_1(\Delta\tilde{u}_n)}$, $V^m_n \equiv \sqrt{\mathrm{var}_1(\Delta\tilde{y}_n)}$ and $V^t_n \equiv V^i_n + V^l_n + V^m_n$.

Now we derive $\mathrm{var}_1(\Delta\tilde{y}_n)$ as follows. By definition,

$$\Delta\tilde{y}_n = \Delta\tilde{x}_n + \Delta\tilde{u}_n, (n = 1, 2, \dots, N). \tag{A.28}$$

Substituting for $\Delta\tilde{x}_n$ according to Eqs. (7) and (12) in Theorem 1, and rearranging yields

$$\Delta\tilde{y}_n = \left[(1+\gamma_n)\beta_n\Delta t_n - \frac{\gamma_n}{\lambda_n}\right](\tilde{s} - \tilde{p}_{n-1}) - \frac{\gamma_n}{\lambda_n}\tilde{p}_{n-1} + \Delta\tilde{u}_n. \tag{A.29}$$

Taking variance on both sides of Eq. (A.29) yields

$$\mathrm{var}_1(\Delta\tilde{y}_n) = \left[(1+\gamma_n)\beta_n\Delta t_n - \frac{\gamma_n}{\lambda_n}\right]^2 \Sigma_{n-1} + \sigma_u^2\Delta t_n + \frac{\gamma_n^2}{\lambda_n^2}(\Sigma_0 - \Sigma_{n-1}). \tag{A.30}$$

Applying Eqs. (A.15) and (A.16) into Eq. (A.30) yields

$$\mathrm{var}_1(\Delta\tilde{y}_n) = \frac{1}{\lambda_n^2}(\Sigma_{n-1} - \Sigma_n) + \frac{\gamma_n^2}{\lambda_n^2}(\Sigma_0 - \Sigma_{n-1}). \tag{A.31}$$

To derive $\mathrm{var}_1(\Delta\tilde{x}_n)$, taking variance of both sides of Eq. (A.28) and rearranging yields

$$\mathrm{var}_1(\Delta\tilde{x}_n) = \mathrm{var}_1(\Delta\tilde{y}_n) - \sigma_u^2\Delta t_n. \tag{A.32}$$

Substituting Eq. (A.30) into Eq. (A.32), using Eq. (A.16) and rearranging yields

$$\mathrm{var}_1(\Delta\tilde{x}_n) = \frac{\sigma_u^2\Delta t_n}{\Sigma_n}(\Sigma_{n-1} - \Sigma_n) + \frac{\gamma_n^2}{\lambda_n^2}(\Sigma_0 - \Sigma_{n-1}). \tag{A.33}$$

By definition, $\mathrm{var}_1(\Delta\tilde{u}_n) = \sigma_u^2\Delta t_n$. Similar calculations yield Eq. (24). Q.E.D.

Proof of Proposition 3. To prove Propositions 3 and 4, we need Lemma 1 as follows:

Lemma 1. *The covariance between the absolute value of two normally distributed variables $\tilde{x}$ and $\tilde{y}$ with correlation $\rho(\tilde{x}, \tilde{y})$ and zero means is equal to*

$$\mathrm{Cov}(|\tilde{x}|, |\tilde{y}|) = \frac{2}{\pi}(\mathrm{Var}(\tilde{x})\mathrm{Var}(\tilde{y}))((1-\rho^2(\tilde{x}, \tilde{y}))^{1/2} - 1 + \rho(\tilde{x}, \tilde{y})\mathrm{ArcSin}[\rho(\tilde{x}, \tilde{y})]). \tag{A.34}$$

Moreover,

$$\mathrm{Cov}(\tilde{x}, \tilde{y}) = 0 \Leftrightarrow \rho(\tilde{x}, \tilde{y}) = 0 \Rightarrow \mathrm{Cov}(|\tilde{x}|, |\tilde{y}|) = 0; \tag{A.35}$$

$$\mathrm{Cov}(\tilde{x}, \tilde{y}) \neq 0 \Leftrightarrow \rho(\tilde{x}, \tilde{y}) \neq 0 \Rightarrow \mathrm{Cov}(|\tilde{x}|, |\tilde{y}|) > 0. \tag{A.36}$$

Given Lemma 1, we prove Proposition 3 as follows. With Eqs. (21) and (A.35), and that noise trading is independent of all other variables, we obtain, for $n = 1, 2, \ldots, N-1$,

$$\begin{aligned}
&\mathrm{Cov}_1(\mathrm{Vol}_n, \mathrm{Vol}_{n+1}) \\
&\quad = \frac{1}{4}\mathrm{Cov}_1(|\Delta\tilde{x}_n| + |\Delta\tilde{u}_n| + |\Delta\tilde{y}_n|, |\Delta\tilde{x}_{n+1}| + |\Delta\tilde{u}_{n+1}| + |\Delta\tilde{y}_{n+1}|) \\
&\quad = \frac{1}{4}[\mathrm{Cov}_1(|\Delta\tilde{x}_n|, |\Delta\tilde{x}_{n+1}|) + \mathrm{Cov}_1(|\Delta\tilde{x}_n|, |\Delta\tilde{y}_{n+1}|) + \mathrm{Cov}_1(|\Delta\tilde{y}_n|, |\Delta\tilde{x}_{n+1}|) \\
&\qquad + \mathrm{Cov}_1(|\Delta\tilde{y}_n|, |\Delta\tilde{y}_{n+1}|)]
\end{aligned} \tag{A.37}$$

Using the condition that noise trading is independent of all other variables again, obtain

$$\begin{aligned}
&\mathrm{Cov}_1(\Delta\tilde{y}_n, \Delta\tilde{y}_{n+1}) = \mathrm{Cov}_1(\Delta\tilde{y}_n, \Delta\tilde{x}_{n+1}) \\
&\quad = \mathrm{Cov}_1(\Delta\tilde{x}_n, \Delta\tilde{y}_{n+1}) = \mathrm{Cov}_1(\Delta\tilde{x}_n, \Delta\tilde{x}_{n+1}).
\end{aligned} \tag{A.38}$$

Given Eqs. (A.37) and (A.38) and the above lemma, we obtain, for $n = 1, 2, \ldots, N-1$,

$$\mathrm{Cov}_1(\Delta\tilde{y}_n, \Delta\tilde{y}_{n+1}) = 0 \Rightarrow \mathrm{Cov}_1(\mathrm{Vol}_n, \mathrm{Vol}_{n+1}) = 0$$

and

$$\mathrm{Cov}_1(\Delta\tilde{y}_n, \Delta\tilde{y}_{n+1}) \neq 0 \Rightarrow \mathrm{Cov}_1(\mathrm{Vol}_n, \mathrm{Vol}_{n+1}) > 0. \tag{A.39}$$

To determine $\mathrm{Cov}_1(\Delta\tilde{y}_n, \Delta\tilde{y}_{n+1})$, we use Eq. (8) and rearrange as follows:

$$\begin{aligned}
\mathrm{Cov}_1(\Delta\tilde{y}_n, \Delta\tilde{y}_{n+1}) &= \mathrm{Cov}_1\left(\frac{1}{\lambda_n}(\Delta\tilde{p}_n - \gamma_n\tilde{p}_{n-1}), \frac{1}{\lambda_{n+1}}(\Delta\tilde{p}_{n+1} - \gamma_{n+1}\tilde{p}_n)\right) \\
&= \frac{-\gamma_{n+1}}{\lambda_n\lambda_{n+1}}\mathrm{Cov}_1(\Delta\tilde{p}_n - \gamma_n\tilde{p}_{n-1}, \Delta\tilde{p}_n + \tilde{p}_{n-1})) \\
&= \frac{-\gamma_{n+1}}{\lambda_n\lambda_{n+1}}(\mathrm{Var}_1(\Delta\tilde{p}_n) - \gamma_n\mathrm{Var}_1(\tilde{p}_{n-1})) \\
&= \frac{-\gamma_{n+1}}{\lambda_n\lambda_{n+1}}((\Sigma_{n-1} - \Sigma_n) - \gamma_n(\Sigma_0 - \Sigma_{n-1}))
\end{aligned} \tag{A.40}$$

The last equality is obtained using Eqs. (3) and (5). Q.E.D.

Proof of Proposition 4. This proof is in the same spirit as the last proof. Given Eqs. (21) and (A.35), we obtain

$$\mathrm{Cov}_1(|\Delta\tilde{p}_n|, \mathrm{Vol}_n) = \tfrac{1}{2}\mathrm{Cov}_1(|\Delta\tilde{p}_n|, |\Delta\tilde{x}_n| + |\Delta\tilde{y}_n|)$$
$$= \tfrac{1}{2}\mathrm{Cov}_1(|\Delta\tilde{p}_n|, |\Delta\tilde{x}_n|) + \tfrac{1}{2}\mathrm{Cov}_1(|\Delta\tilde{p}_n|, |\Delta\tilde{y}_n|). \quad \text{(A.41)}$$

Using the condition that noise trading is independent to all other variables again, obtain

$$\mathrm{Cov}_1(\Delta\tilde{p}_n, \Delta\tilde{x}_n) = \mathrm{Cov}_1(\Delta\tilde{p}_n, \Delta\tilde{y}_n)$$
$$= \mathrm{Cov}_1\left(\Delta\tilde{p}_n, \frac{1}{\lambda_n}(\Delta\tilde{p}_n - \gamma_n\tilde{p}_{n-1})\right)$$
$$= \frac{1}{\lambda_n}\mathrm{Var}_1(\Delta\tilde{p}_n) > 0. \quad \text{(A.42)}$$

Eqs. (A.36), (A.41) and (A.42) together imply $\mathrm{Cov}_1(|\Delta\tilde{p}_n|, \mathrm{Vol}_n) > 0$. Q.E.D.

Proof of Theorem 2. Let

$$q_n \equiv \alpha_n\lambda_n, \quad \text{(A.43)}$$

$$z_n \equiv \omega_n\lambda_n. \quad \text{(A.44)}$$

Plugging Eq. (A.44) into Eq. (14) yields

$$\gamma_n = 1 - K - z_n. \quad \text{(A.45)}$$

Plugging Eqs. (A.43) and (A.45) into Eq. (16) yields

$$\alpha_{n-1} = \frac{(2 - K - z_n)^2}{4\lambda_n(1 - q_n)}. \quad \text{(A.46)}$$

Multiplying both sides of Eq. (A.46) by λ_{n-1} and using Eq. (A.43) yields

$$\frac{\lambda_n}{\lambda_{n-1}} = \frac{(2 - K - z_n)^2}{4(1 - q_n)q_{n-1}}. \quad \text{(A.47)}$$

Multiplying both sides of Eq. (17) by λ_n and using Eqs. (A.44) and (A.45) yields

$$\omega_{n-1} = \frac{(1 - K)(2 - K - z_n)}{\lambda_n}. \quad \text{(A.48)}$$

Multiplying both sides of Eq. (A.48) by λ_{n-1} and using Eq. (A.44) yields

$$\frac{\lambda_n}{\lambda_{n-1}} = \frac{(1 - K)(2 - K - z_n)}{z_{n-1}}. \quad \text{(A.49)}$$

Equating Eq. (A.47) with Eq. (A.49) yields

$$z_{n-1} = \frac{4(1-K)(1-q_n)}{(2-K-z_n)} q_{n-1}. \tag{A.50}$$

Assume that all time intervals between auctions are equally spaced, i.e.,

$$\Delta t_n = 1/N. \tag{A.51}$$

Plugging Eqs. (A.43) and (A.51) into Eq. (11) yields

$$\beta_n = \frac{N(1-2q_n)}{2\lambda_n(1-q_n)}. \tag{A.52}$$

Plugging Eqs. (A.45) and (A.51) into Eq. (12) yields

$$\theta_n = \frac{-N(1-K-z_n)}{\lambda_n}. \tag{A.53}$$

Plugging Eqs. (A.45), (A.52) and (A.53) into Eq. (13) and rearranging yields

$$\lambda_n = \sqrt{\frac{N(K+z_n-2q_n)\Sigma_n}{2(1-q_n)\sigma_u^2}}. \tag{A.54}$$

From Eq. (A.54), we obtain

$$\frac{\lambda_n^2}{\lambda_{n-1}^2} = \frac{(K+z_n-2q_n)(1-q_{n-1})\Sigma_n}{(K+z_{n-1}-2q_{n-1})(1-q_n)\Sigma_{n-1}}. \tag{A.55}$$

Plugging Eqs. (A.43) and (A.45) into Eq. (15) and rearranging yields

$$\Sigma_n = \frac{(2-K-z_n)}{2(1-q_n)} \Sigma_{n-1}. \tag{A.56}$$

Plugging Eq. (A.56) into Eq. (A.55) yields

$$\frac{\lambda_n^2}{\lambda_{n-1}^2} = \frac{(K+z_n-2q_n)(1-q_{n-1})(2-K-z_n)}{2(K+z_{n-1}-2q_{n-1})(1-q_n)^2}. \tag{A.57}$$

Squaring both sides of Eq. (A.47) yields

$$\frac{\lambda_n^2}{\lambda_{n-1}^2} = \frac{(2-K-z_n)^4}{16(1-q_n)^2 q_{n-1}^2}. \tag{A.58}$$

Equating Eq. (A.57) with Eq. (A.58), using Eq. (A.50) and rearranging terms yields Eqs. (28), (30) and (31). From Eqs. (20), (A.19) and (A.20), we obtain the inequality Eq. (29).

Finally, we show that the cubic Eq. (28) has a unique root satisfying the inequality condition (29). To do so, we rewrite Eq. (28) as

$$f(q_{n-1}) = g(q_{n-1}), \tag{A.59}$$

where

$$f(q_{n-1}) = b_n + a_n q_{n-1}, \tag{A.60}$$

$$g(q_{n-1}) = q_{n-1}^2(1 - q_{n-1}). \tag{A.61}$$

Given the boundary condition $q_N = z_N = 0$, both sides of Eq. (A.59) are functions of q_{n-1}. It is straightforward to show that the two functions, $f(q_{n-1})$ and $g(q_{n-1})$ have a unique point of intersection in the interval specified by Eq. (29). Moreover, if $1 < K < 2$, then simple calculations show that the unique root lies in a smaller interval, $q_{n-1} \in (0, 0.5)$. Q.E.D.

Proof of Proposition 5. Given the limiting result of the liquidity parameter, i.e., $\lambda_n \to \infty$, as $\sigma_u^2 \to 0$, $\forall n$, as discussed in the main text, all other limiting results of the proposition are straightforward to obtain except the price adjustment process in Eq. (43). Hence, we only prove Eq. (43) in what follows and leave the rest to the readers. Substituting $\Delta \tilde{x}_n$ from Eq. (7) into Eq. (8) and applying Theorem 2 for parameters β_n, θ_n, λ_n, γ_n, and Σ_n yields

$$\begin{aligned}
\Delta \tilde{p}_n &= \gamma_n \tilde{p}_{n-1} + (1+\gamma_n)\lambda_n \beta_n \Delta t_n(\tilde{s} - \tilde{p}_{n-1}) + \lambda_n \theta_n \Delta t_n \tilde{s} + \lambda_n \Delta \tilde{u}_n \\
&= \frac{K + z_n - 2q_n}{2(1-q_n)}(\tilde{s} - \tilde{p}_{n-1}) + \left(\frac{(K + z_n - 2q_n)\Sigma_n}{2(1-q_n)}\right)^{1/2}\left(\frac{\Delta \tilde{u}_n}{\sqrt{\sigma_u^2 \Delta t_n}}\right) \\
&= \left(1 - \frac{\Sigma_n}{\Sigma_{n-1}}\right)(\tilde{s} - \tilde{p}_{n-1}) + \left(\left(1 - \frac{\Sigma_n}{\Sigma_{n-1}}\right)\Sigma_n\right)^{1/2}\tilde{\xi}_n \\
&= c_n(\tilde{s} - \tilde{p}_{n-1}) + (c_n \Sigma_n)^{1/2}\tilde{\xi}_n, \text{ where } c_n = 1 - \frac{\Sigma_n}{\Sigma_{n-1}} \text{ and } \tilde{\xi}_n \sim \mathrm{N}(0, 1).
\end{aligned} \tag{A.62}$$

Note that $\tilde{\xi}_n$ is a standard normal random variable, which does not depend on the variance of noise trading, σ_u^2. By the definition in Eq. (4), the error variance of price is strictly decreasing over time, i.e., $0 < \Sigma_n < \Sigma_{n-1}$, $\forall n$. In addition, Eq. (33) indicates that the ratio Σ_n/Σ_{n-1} depends only on the two parameters q_n and z_n, both of which are uniquely determined given a finite number of period N. Hence, the speed of the price adjustment, $c_n = \Sigma_n/\Sigma_{n-1}$, in each period is a constant such that $0 < c_n < 1$, $\forall n$, and the constant does not depend on the variance of noise trading, σ_u^2, either. Consequently, the price adjustment process does not depend on the variance of noise trading, σ_u^2. Q.E.D.

References

Admati, A., Pfleiderer, P., 1988. A theory of intraday patterns: volume and price variability. Review of Financial Studies 1, 3–40.

Alpert, M., Raiffa, H., 1959. A progress report on the training of probability assessors, Judgment under uncertainty: heuristics and biases. Kahneman, D., Slovic, P., Tversky, A. (Eds.), Cambridge University Press, Cambridge.

Aumann, R.J., 1976. Agreeing to disagree. Annals of Statistics 4, 1236–1239.

Campbell, J.Y., Grossman, S.J., Wang, J., 1993. Trading volume and serial correlation in stock returns. Quarterly Journal of Economics 108, 905–939.

De Long, J.B., Shleifer, A., Summers, L.H., Waldmann, R.J., 1990. Noise trader risk in financial markets. Journal of Political Economy 98, 703–738.

Foster, F.D., Viswanathan, S., 1990. A theory of the interday variations in volume, variance, and trading costs in securities markets. Review of Financial Studies 3, 593–624.

Foster, F.D., Viswanathan, S., 1993. The effect of public information and competition on trading volume and price volatility. Review of Financial Studies 6, 23–56.

Gallant, A.R., Rossi, P.E., Tauchen, G., 1992. Stock prices and volume. Review of Financial Studies 5, 199–242.

Gerety, M.S., Mulherin, H.J., 1992. Trading halts and market activity: an analysis of volume at the open and the close. Journal of Finance 47, 1765–1784.

Grossman, S.J., Stiglitz, J.E., 1980. On the impossibility of informationally efficient markets. American Economic Review 70, 393–408.

Grundy, B.D., McNichols, M., 1989. Trade and revelation of information through prices and direct disclosure. Review of Financial Studies 2, 495–526.

Harris, L., 1987. Transaction data tests of the mixture of distribution hypothesis. Journal of Financial and Quantitative Analysis 22, 127–141.

Harris, M., Raviv, A., 1993. Differences of opinion make a horse race. Review of Financial Studies 6, 473–506.

Harrison, J.M., Kreps, D.M., 1978. Speculative investor behavior in a stock market with heterogeneous expectations. Quarterly Journal of Economics 93, 323–336.

He, H., Wang, J., 1995. Differential information and dynamic behavior of stock trading volume. Review of Financial Studies 8, 919–972.

Hindy, A., 1989. An equilibrium model of future market dynamics. Working paper, MIT, Cambridge.

Holden, C.W., Subrahmanyam, A., 1992. Long-lived private information and imperfect competition. Journal of Finance 47, 247–270.

Jain, P.C., Joh, G.H., 1988. The dependence between hourly prices and trading volume. Journal of Financial and Quantitative Analysis 23, 269–284.

Kahneman, D., Tversky, A., 1982. Intuitive prediction: biases and corrective procedures. In: Kahneman, D., Slovic, P., Tversky, A. (Eds.), Judgment under uncertainty: heuristics and biases. Cambridge University Press, Cambridge.

Kandel, E., Pearson, N.D., 1992. Differential interpretation of public signals and trade in speculative markets. Working paper, University of Rochester, Rochester.

Karpoff, J.M., 1987. The relation between price changes and trading volume: a survey. Journal of Financial and Quantitative Analysis 22, 109–126.

Kim, O., Verrecchia, R.E., 1991. Market reactions to anticipated announcements. Journal of Financial Economics 30, 273–309.

Kyle, A.S., 1985. Continuous auctions and insider trading. Econometrica 53, 1315–1336.

Kyle, A.S., Wang, F.A., 1997. Speculation duopoly with agreement to disagree: Can overconfidence survive the market test? Journal of Finance 52, 2073–2090.

Oskamp, S., 1965. Overconfidence in case study judgments. Journal of Consulting Psychology 29, 261–265.

Shalen, C.T., 1993. Volume, volatility, and the dispersion of beliefs. Review of Financial Studies 6, 405–434.
Varian, H.R., 1985. Divergence of opinion in complete markets. Journal of Finance 40, 309–317.
Varian, H.R., 1989. Differences of opinion in financial markets. In: Stone, C.C. (Ed.), Financial risk: theory, evidence and implications, Federal Reserve Bank of St. Louis, Kluwer, Boston, pp. 3–37.
Wang, F.A., 1997. Overconfidence, delegated fund management, and survival. Working paper, Columbia University, New York.
Wang, J., 1994. A Model of competitive stock trading volume. Journal of Political Economy 102, 127–168.
Wood, R.A., McInish, T.H., Ord, J.K., 1985. An investigation of transaction data for NYSE stocks. Journal of Finance 40, 723–741.